Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 8: Mathematical Expectation

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Mathematical Expectation

1)The Expected Value of a Random Variable

- **Expectation** is the name given to the process of averaging when a random variable is involved. The **Mean value**, **the Statistical average**, or the **Expected Value** of a random variable **X** are different terms for the **Expectation** and denoted as $E(X)$or $\bar{X}$
- Mathematical expectation can be thought of more or less as an **average over the long run.**
- The expectation of **X** is very often called the **mean** of **X** and is denoted by μ_X, or simply μ and it is often called a **measure of central tendency.**

✓ **Definition 1. Expected Value.**

- If **X** is a discrete random variable and **f(x)** is the value of its probability mass function at **x**, the expected value of **X** is

$$E(X) = \sum_x x \cdot f(x)$$

- if **X** is a continuous random variable and **f(x)** is the value of its probability density at **x**, the expected value of **X** is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

The **sum** or the **integral** exists; otherwise, the mathematical expectation is **undefined.**

✓ **Example 1**: the probability function of **X** is given in tabular form

x	0	1	2
$f(x)$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

The mathematical expectation is

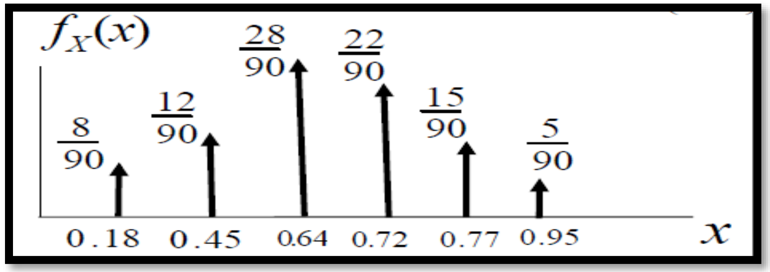
$$E(X) = 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22} = \frac{1}{2}$$

✓ **Example 2**: (**Discrete Random Variable**)

Ninety people are randomly selected and the fractional dollar value of coins in their pockets is counted. If the count goes above a dollar, the dollar value is discarded and only the portion from **0₵** to **99₵** is accepted. It is found that **8, 12, 28, 22, 15,** and **5** people had **18₵, 45₵, 64₵, 72₵, 77₵, and 95₵** in their pockets, respectively. Find the average of these values.

Solution:

$$E[X]=\overline{X}=\sum_{i=1}^{6} x_i\, P(x_i)=0.18\left(\frac{8}{90}\right)+0.45\left(\frac{12}{90}\right)+0.64\left(\frac{28}{90}\right)+0.72\left(\frac{22}{90}\right)+0.77\left(\frac{15}{90}\right)+0.95\left(\frac{5}{90}\right)=0.632\ \$$$



✓ **Example 3: (Continuous Random Variable)**

Find the mean value for the exponential random variable

$$f_X(x)=\begin{cases}\frac{1}{b}e^{-(x-a)/b} & x>a\\ 0 & x<a\end{cases}$$

Solution:

$$E[X]=\overline{X}=\int_{-\infty}^{+\infty} x\, f_X(x)dx=\int_{a}^{\infty} x\frac{1}{b}e^{-(x-a)/b}dx$$
$$=a+b$$

2)Expected Value for a Function of a Random Variable

- ✓ There are many problems, in which we are interested not only in the expected value of a random variable **X**, but also in the expected values of random variables related to **X**. Thus, we might be interested in the random variable **Y**, whose values are related to those of **X** by means of the equation **$y = g(x)$;** to simplify our notation, we denote this random variable by **$g(X)$.**
- ✓ Let **X** be a discrete random variable with probability function **$f(x)$.** Then **$Y = g(X)$** is also a discrete random variable:
- ✓ **Theorem 1**.
  - If **X** is a discrete random variable and **$f(x)$** is the value of its probability mass function at **x**, the expected value of a real function **$g(X)$** is given by

$$E[g(X)] = \sum_{x} g(x) \cdot f(x)$$

  - If **X** is a continuous random variable and **$f(x)$** is the value of its probability density at **x**, the expected value of **$g(X)$** is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$$

✓ **Example 4**: If $\mathbf{X}$ is the number of points rolled with a balanced die, find the expected value of $\mathbf{g(X) = 2X^2 + 1}$.

Solution: Since each possible outcome has the probability $\mathbf{1/6}$, we get:

$$E[g(X)] = \sum_{x=1}^{6} (2x^2 + 1) \cdot \frac{1}{6}$$
$$= (2 \cdot 1^2 + 1) \cdot \frac{1}{6} + \cdots + (2 \cdot 6^2 + 1) \cdot \frac{1}{6}$$
$$= \frac{94}{3}$$

✓ **Example 5**: If $\mathbf{X}$ has the probability density

$$f(x) = \begin{cases} e^x & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of $\mathbf{g(X) = e^{3X/4}}$.

Solution:

$$E[e^{3X/4}] = \int_0^{\infty} e^{3x/4} \cdot e^{-x}\, dx$$
$$= \int_0^{\infty} e^{-x/4}\, dx$$
$$= 4$$

✓ mathematical expectations theorems which enable us to calculate expected values from other known or easily computed expectations

✓ **Theorem 2**. If **a** and **b** are constants, then

$$E(aX+b) = aE(X)+b$$

✓ **Example 6**: Making use of the fact that

$$E(X^2) = (1^2+2^2+3^2+4^2+5^2+6^2)\cdot\frac{1}{6} = \frac{91}{6}$$

For the random variable of **example 4**, rework that example.

$$E(2X^2+1) = 2E(X^2)+1 = 2\cdot\frac{91}{6}+1 = \frac{94}{3}$$

✓ **Example 7**: If the probability density of **X** is given by

$$f(x) = \begin{cases} 2(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Show that

$$E(X^r) = \frac{2}{(r+1)(r+2)}$$

(b) And use this result to evaluate

$$E[(2X+1)^2]$$

(a)

$$E(X^r) = \int_0^1 x^r \cdot 2(1-x)\,dx = 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = \frac{2}{(r+1)(r+2)}$$

(b) Since $E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1$ and substitution of $r = 1$ and $r = 2$ into the preceding formula yields $E(X) = \frac{2}{2\cdot 3} = \frac{1}{3}$ and $E(X^2) = \frac{2}{3\cdot 4} = \frac{1}{6}$, we get

$$E[(2X+1)^2] = 4\cdot\frac{1}{6} + 4\cdot\frac{1}{3} + 1 = 3$$

✓ **Theorem 3:** If **X** and **Y** are any random variables, then

$$\mathbf{E(X+Y) = E(X) + E(Y)}$$

✓ **Theorem 4:** If **X** and **Y** are **independent random variables**, then

$$\mathbf{E(XY) = E(X)E(Y)}$$

- ✓ The concept of a mathematical expectation can be extended to situations involving more than one random variable. For instance, if **Z** is the random variable whose values are related to those of the two random variables **X** and **Y** by means of the equation **z = g(x, y),** we can state the following theorem.
- ✓ **Theorem 5:**
  - If **X** and **Y** are discrete random variables and **f (x, y)** is the value of their joint probability distribution at **(x, y),** the expected value of **g(X, Y)** is

$$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) \cdot f(x,y)$$

  - If **X** and **Y** are continuous random variables and **f (x, y)** is the value of their joint probability density at **(x, y),** the expected value of **g(X, Y)** is

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

  - **Generalization** of this theorem to functions of any finite number of random variables is **straightforward**.

✓ **Example 8**: Given the joint probability table, Find the expected value of $g(X, Y) = X + Y$.

			x	
		0	1	2
	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
y	1	$\frac{2}{9}$	$\frac{1}{6}$	
	2	$\frac{1}{36}$		

- **Solution**

$$E(X+Y) = \sum_{x=0}^{2}\sum_{y=0}^{2}(x+y)\cdot f(x,y)$$

$$= (0+0)\cdot\frac{1}{6} + (0+1)\cdot\frac{2}{9} + (0+2)\cdot\frac{1}{36} + (1+0)\cdot\frac{1}{3}$$

$$+ (1+1)\cdot\frac{1}{6} + (2+0)\cdot\frac{1}{12}$$

$$= \frac{10}{9}$$

✓ **Example 9**: If the joint probability density of **X** and **Y** is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \text{for } 0 < x < 1, 1 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of $g(X,Y) = X/Y^3$.

- **Solution**

$$E(X/Y^3) = \int_1^2 \int_0^1 \frac{2x(x+2y)}{7y^3}\,dx\,dy$$

$$= \frac{2}{7}\int_1^2 \left(\frac{1}{3y^3} + \frac{1}{y^2}\right) dy$$

$$= \frac{15}{84}$$